COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **20** (2) (2000) 257–262

ON SIZE RAMSEY NUMBERS OF GRAPHS WITH BOUNDED DEGREE

VOJTĚCH RÖDL and ENDRE SZEMERÉDI

Received December 21, 1998

Answering a question of J. Beck [2], we prove that there exists a graph G on n vertices with maximum degree three and the size Ramsey number $\hat{r}(G) \geq cn(\log n)^{\alpha}$ where α and c are positive constants.

For graphs G and F, write $F \rightarrow G$ to mean that if the edges of F are colored by red and blue, then F contains a monochromatic copy of G.

Erdős, Faudree, Rousseau and Schelp [4] were the first to consider the question of how few edges F can have and still $F \rightarrow G$. Following [4], by the size Ramsey number $\hat{r}(G)$, we mean the least integer $\hat{r}$ such that there exists a graph F with $\hat{r}$ edges for which $F \rightarrow G$, i. e., $\hat{r}(G) = min\{|F| : F \rightarrow G\}$ (where $|F|$ denotes the cardinality of the edge set of F).

Clearly, $\hat{r}(K_{1,n}) = 2n-1$, where $K_{1,n}$ denotes the star on $n+1$ vertices. In [1], J. Beck answered a question of P. Erdős [3], proving that there exists an absolute constant c ($c \leq 900$) such that

$$\hat{r}(P_n) < cn. \tag{1}$$

Later in [2], J. Beck raised the following question, which if answered affirmatively, would give a far reaching generalization to (1).

The first author is partly supported by the NSF grant DMS 9704114 and the grant MR1-181 of the Cooperative Grant Program of the Civilian Research and Development Foundation.

The second author is partly supported by the NSF grant DMS 9801396.

Mathematics Subject Classification (1991):

Problem. Let $G_{n,r}$ be a graph with n vertices and maximum degree r. Decide whether $\hat{r}(G_{n,r})<c(r)n$ where the constant $c(r)$ depends only on r. This has shown to be true when $G_{n,r}$ is a cycle [6] or a tree [5].

The aim of this note is to answer Beck's question showing that the statement above fails already for $r=3$. More precisely, we prove the following theorem:

Theorem 1. *There exists positive constants c and α, and a graph $G=(V,E)$ with $|V|=n$ and maximum degree, $\Delta(G)=3$ such that*

$$\hat{r}(G) \geq cn(\log_2 n)^{\alpha}. \tag{2}$$

Proof. . Since we believe that the lower bound (2) is quite far from the best possible, we will make no effort to find the best possible c and α. We will prove that for $n\geq n_0$, (2) holds with $c=\frac{1}{10}$ and $\alpha=\frac{1}{60}$.
For n sufficiently large, choose m of the form $m=2^{t-1}$ where $t\geq 3$ is an integer, so that

$$2\frac{\log_2 n}{\log_2\log_2 n} \leq m \leq 4\frac{\log_2 n}{\log_2\log_2 n}.$$

Construction of G. Consider a binary tree B with $1+2+\cdots+2^{t-1}$ vertices and $L(B)$ be the set of all leafs. Let B_1 and B_2 be two disjoint copies of B with roots x_1 and x_2 respectively. Let T be a tree defined by

$$V(T)=V(B_1)\cup V(B_2)\cup\{y_1,y_2\}$$

and

$$E(T)=E(B_1)\cup E(B_2)\cup\{x_1,y_1\}\cup\{y_1,y_2\}\cup\{x_2,y_2\}.$$

That is, T is a binary tree with 2^{t+1} vertices and rooted edge $\{y_1,y_2\}$. Let $L(T)=L(B_1)\cup L(B_2)$ be a set of all leafs of T.

Finally, consider *a* cycle $H\simeq C_{2m}$ of length $2m$ and the vertex set $V(H)=L(T)$.

Considering labeled vertices of $L(T)$ there are clearly $\frac{(2m)!}{2m}$ of such cycles H. Label them H_i for $i=1,2,\dots,\frac{(2m)!}{2m}$. For each H_i, let T_{H_i} be a copy of T so that T_{H_i}'s are vertex disjoint for distinct i's. For each $i\leq\frac{(2m)!}{2m}$ we also set $\tilde{H}_i=H_i\cup T_{H_i}$.

Let $\mathcal{H}$ be the set of all graphs $\tilde{H}_i$ constructed above. Note that for each $\tilde{H}_i\in\mathcal{H}$ any automorphism φ of $\tilde{H}_i$ satisfies $\varphi(E(T_{H_i}))=E(T_{H_i})$. This is due to the fact that the only vertices of degree two in $\tilde{H}_i$ are y_1 and y_2 , so they must be mapped to $\{y_1,y_2\}$. So, x_1 and x_2 must be mapped to $\{x_1,x_2\}$ since they can't be mapped to $\{y_1,y_2\}$ and if they were mapped anywhere else then they wouldn't be adjacent to $\varphi(y_1)$ and $\varphi(y_2)$ respectively. Similarly a vertex on any level of the tree must be mapped by φ to a vertex on that same level. Thus, in particular, any automorphism of $\tilde{H}_i$ is an automorphism of T_{H_i}. On the other hand, there are

$$2^1 \cdot 2^2 \cdot 2^{2^2} \cdot \ldots \cdot 2^{2^{t-1}} < 2^{2^t} = 2^{2m}$$

automorphisms of $T_{H_i} \simeq T_H$ and hence there are at least

$$\left(\frac{(2m)!}{2m}\right) \div \left(2^{2m}\right) = \frac{(2m-1)!}{2^{2m}} > m^m$$

distinct automorphism types among the graphs $\tilde{H}_i \in \mathcal{H}$.

Observe that

$$m^m > \left(\frac{\log_2 n}{\log_2 \log_2 n}\right)^{\frac{2\log_2 n}{\log_2 \log_2 n}} = 2^{2\log_2 n(1 - \frac{\log_2 \log_2 \log_2 n}{\log_2 \log_2 n})} > n > q = \lfloor \frac{n}{4m} \rfloor$$

and without loss of generality assume that $\{\tilde{H}_1, \tilde{H}_2, \ldots \tilde{H}_q\}$ are pairwise non-isomorphic.

We define G to be a disjoint union of $\tilde{H}_i, 1 \le i \le q$. Note that G has at most n vertices and maximum degree 3 while the minimum degree is two. This means that

$$\alpha(G) \le \frac{3n}{5}. \tag{3}$$

Set $l = \frac{1}{10} 2^{(\frac{1}{15m}\log_2 n)} = \frac{1}{10} n^{\frac{1}{15m}}$ and observe that

$$n^{\frac{1}{15m}} \ge 2^{\frac{\log_2 \log_2 n}{60 \log_2 n} \log_2 n} \ge (\log_2 n)^{\frac{1}{60}}.$$

Note that Theorem 1 immediately follows from the following.
Fact. *If F is a graph of nl edges, then $F \not\longrightarrow G$.*

□

Proof of Fact. Set $k = 10l$. Let $V_{high} = \{v \in V(F), deg(v) > k\}$. Since $2nl \ge \sum\{deg(v), v \in V_{high}\}$ we infer that

$$|V_{high}| < \frac{n}{5}.$$

Now we focus on vertices and edges spanned by the set $V_{low} = V - V_{high}$. For a set $X \subset V_{low}$ let $F[X]$ be a subgraph of F induced by X.

We say that *an edge $e \subset V_{low}$ can see a $2m$ element set $S \subset V_{low}$* if there exists a set $R \subset V_{low}, |R| = 2(1+2+\ldots 2^{t-2})$ and edge preserving $1-1$

mapping (isomorphism into) $\varphi : T \longrightarrow F[e \cup R \cup S]$ such that $\varphi(\{y_1, y_2\}) = e$ and $\varphi(L(T)) = S$
If moreover H is a cycle of length $2m, V(H) \subset V_{low}$ and for some $i = 1, 2, \ldots q$, φ is a $1-1$, edge preserving mapping

$$\varphi : \tilde{H}_i \longrightarrow F[e \cup R \cup V(H)]$$

such that $\varphi(\{y_1, y_2\}) = e$ and $\varphi(L(T)) = V(H)$ we will say that e *can see* H *by* i.

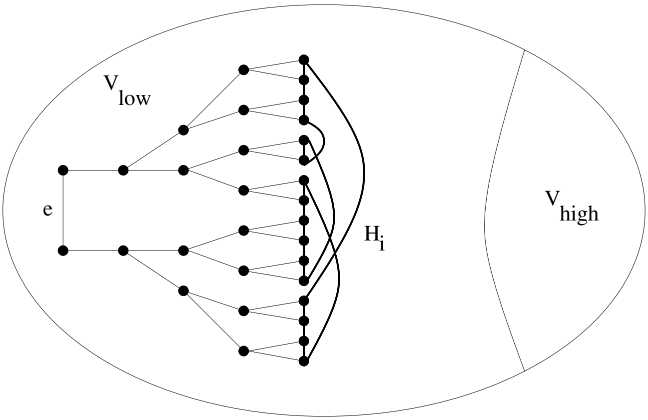

Fig. 1. The edge e can see H by i.

Fix an arbitrary edge $e \subset V_{low}$. Due to the fact that $deg(v) \le k$ for each $v \in V_{low}$, there are at most

$$k^2 \binom{k}{2}^{2(1+2+\ldots+2^{t-2})} \le k^{8m}$$

sets of size $2m$ that can be seen from e. Since there are no more than k^{2m} cycles C_{2m} spanned on any $2m$ element set $S \subset V(F)$, we infer that edge e can see by some i at most

$$k^{8m} k^{2m} = k^{10m}$$

cycles C_{2m}.

Set $E_{low} = E(F[V_{low}])$ and consider an auxilliary bipartite graph Γ with vertex set

$$V(\Gamma) = E_{low} \cup \{1, 2, \ldots, q\}.$$

For $e \in E_{low}$ and $i \in \{1,2,\dots q\}$ we set $\{e,i\} \in E(\Gamma)$ if there exists a cycle H of length $2m$, $V(H) \subset V_{low}$ so that e can see H by i.

Since $deg_{\Gamma}(e) \leq k^{10m}$ and $|E_{low}| \leq nl$, there exists i_0 with

$$\deg_{\Gamma}(i_0) \leq \frac{nlk^{10m}}{q} \leq 5mlk^{10m} \leq 20\frac{\log_2 n}{\log_2 \log_2 n}(\log_2 n)\, n^{\frac{10m}{15m}} = o(n).$$

In other words, there exists $i_0 \in \{1,2,\dots q\}$ so that the set $N_{\Gamma}(i)$ of Γ-neighbours of i_0 (i.e. edges which can see a cycle $H \simeq C_{2m}$ by i_0) has cardinality $o(n)$.

Coloring F. *We color $N_{\Gamma}(i_0)$ together with all edges incident to V_{high} red. All remaining edges are colored blue.*

Let $F = F^{red} \cup F^{blue}$ be a coloring described above and let $\tau(F^{red}) = Min\{|W| : W \subseteq V(F^{red})$ and $e \cap W \neq 0$ for all $e \in E(F^{red})\}$. Observe that $\tau(F^{red}) \leq \frac{n}{5} + o(n) < \frac{2n}{5}$ while $\tau(G)$, the minimum number of vertices of G which are incident to all edges of G satisfies $\tau(G) = |V(G)| - \alpha(G) = n - \alpha(G) \geq \frac{2n}{5}$. Consequently, $\tau(F^{red}) < \tau(G)$.

Suppose that there is a red copy G^{red} of G. This would, however, mean that $\tau(G^{red}) \leq \tau(F^{red})$ which is a contradiction.

On the other hand, since all edges that can see a cycle of length $2m$ by i_0 are colored red, $\tilde{H}_{i_0}$ and consequently G are not subgraphs of F^{blue}.

□

Concluding Remark. Set $\hat{r}(n,\Delta) = Max_G \hat{r}(G)$, where the maximum is taken over all graphs G with n vertices and maximum degree Δ. We conjecture that for any $\Delta \geq 3$ there is $\epsilon > 0$ such that

$$n^{1+\epsilon} \leq \hat{r}(n,\Delta) \leq n^{2-\epsilon}.$$

Acknowledgement. We thank Jason Hunt for many comments which improved the manuscript.

References

[1] J. Beck: On size Ramsey number of paths, trees and circuits I, *J. Graph Theory,* **7** (1983), 115–129.

[2] J. Beck: On Size Ramsey Number of Paths, Trees and Circuits II, in: *Mathematics of Ramsey Theory* (J. Nešetřil and V. Rödl eds.), 34–45, Springer-Verlag, 1990.

[3] P. Erdős: On the combinatorial problems which I would most like to see solved, *Combinatorica* **1** (1981), 25–42.

[4] P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp: The size Ramsey number, *Period. Math. Hungar.,* **9** (1978), 145–161.

[5] J. Friedman and N. Pippenger: Expanding graphs contain all small trees, *Combinatorica* **7** (1981), 71–76.

[6] P. E. Haxell, Y. Kohayahawa and T. Łuczak: The induced size-Ramsey number of cycles, *Combin. Probab. Comput.*, **4** (1995), 217–239.

Vojtěch Rödl

Emory University,
Atlanta, GA 30322, U.S.A.
rodl@mathcs.emory.edu

Endre Szemerédi

Rutgers University,
Piscataway, NJ 08854, U.S.A.
szemered@cs.rutgers.edu